Algebraic Geometry: Lines on a Quadric Surface

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Projective Quadric Surface

A nonsingular projective quadric surface

$$Q = V(xy - zw)$$

is the set of all solutions to the equation $xy - zw = 0$ in $\mathbb{P}^3$.

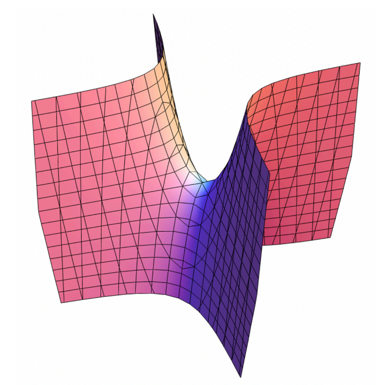
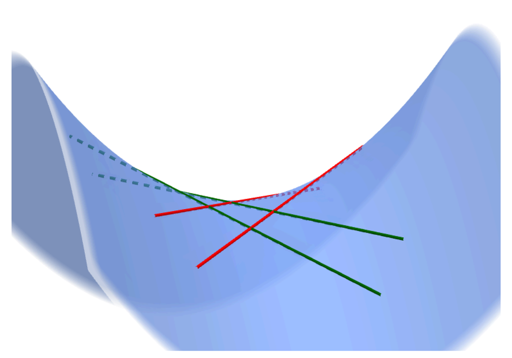


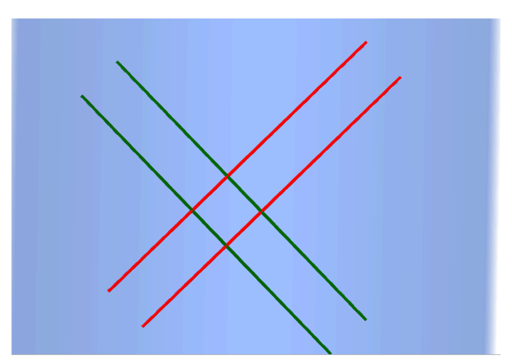
Figure 1. A 3-dimensional 'face' of a projective quadric surface.

Fundamental Property of a quadric Surface Q

Any nonsingular quadric surface Q in $\mathbb{P}^3$ contains two disjoint families $\mathcal{F}_1$ and $\mathcal{F}_2$ of lines such that

1. no two distinct lines from the same family intersect,
$$\ell_1 \cap \ell_2 = \emptyset \quad \text{for any } \ell_1,\ \ell_2 \in \mathcal{F}_1 \text{ or any } \ell_1,\ \ell_2 \in \mathcal{F}_2$$
2. two lines from different families intersect at a single point.
$$\ell_1 \cap \ell_2 = \{\text{a point}\} \quad \text{for } \ell_1 \in \mathcal{F}_1,\ \ell_2 \in \mathcal{F}_2$$
3. The union of lines from each family cover the quadric surface S completely.
$$\bigcup_{\ell \in \mathcal{F}_1} \ell = Q = \bigcup_{m \in \mathcal{F}_2} m$$





The Families in More Specific Terms

$$\mathcal{F}_1 = \{\ell_r \mid r \in \mathbb{C}\} \cup \{n\}, \qquad \mathcal{F}_2 = \{n_c \mid c \in \mathbb{C}\} \cup \{\ell\}$$

where

$$\ell_r = V(w - rx,\ y - rz), \qquad n = V(x, z)$$
$$n_c = V(x - cz,\ w - cy), \qquad \ell = V(y, z)$$

Finding the Lines: A Brief Methodology

We choose an arbitrary point on the quadric surface and find the tangent space at that point. The intersection of the tangent space with the surface itself gives us two distinct lines on the surface. From these two lines, we can define entire families of lines with specific characteristics.

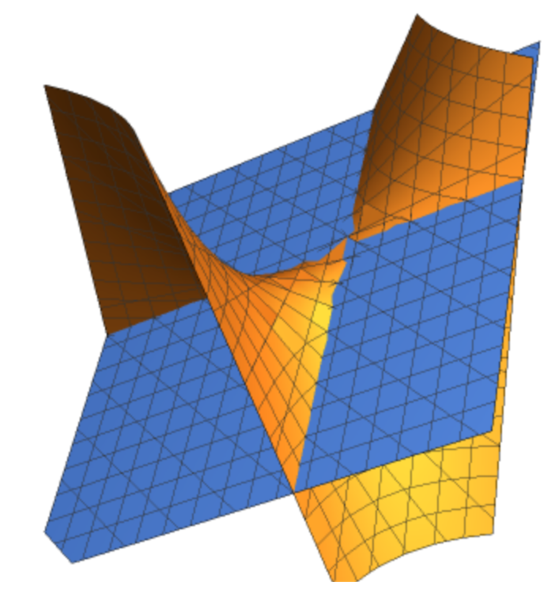


Figure 2. A tangent plane intersecting Q

Generalizing the Families

It can be shown that there exists a suitable change of coordinate basis that transforms the standard quadric surface into any other quadric surface and vice-versa. Therefore, the 'grid-structure' that these lines have is preserved in all quadric surfaces not only the Standard one, though the specific forms of the lines will vary.

Lines in a Projective Cubic Surface?

We performed a similar process on the Fermat cubic surface

$$S = V(x_0^3 + x_1^3 + x_2^3 + x_3^3)$$

and showed that the cubic surface also has lines in it, however, there are only 27 distinct lines, rather than families of infinite lines like in the quadric case. The lines in the cubic surface have no such 'grid-structure' as described for the quadric surface.

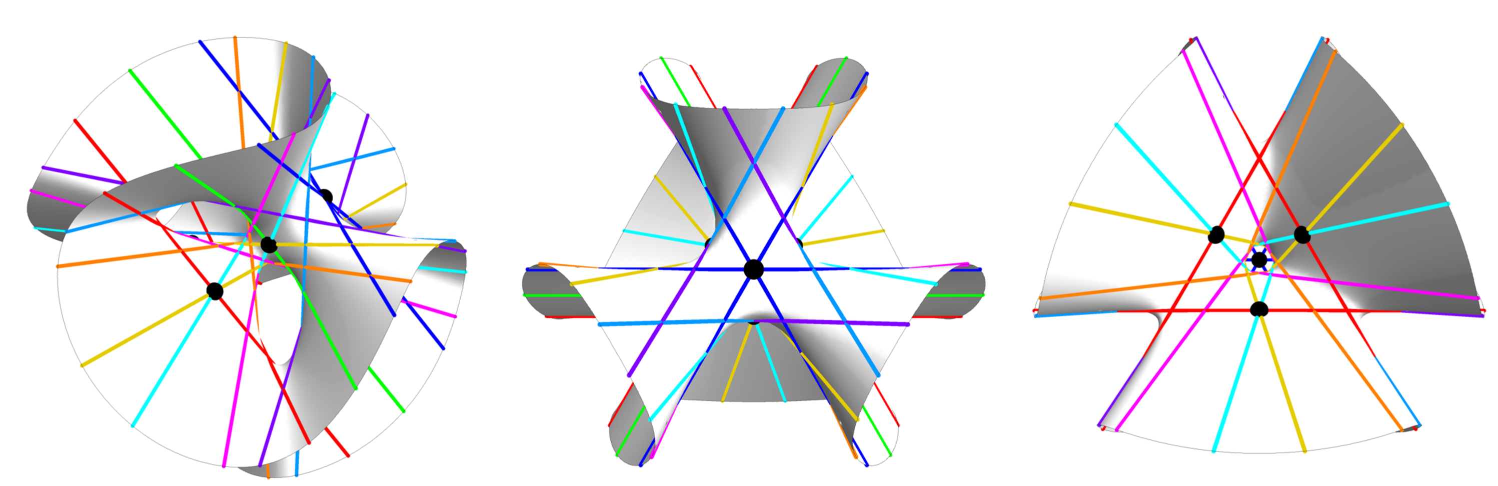


Figure 3. The Clebsch Cubic; a cubic with special property that all 27 lines are real providing a unique opportunity for visualization.

The Plücker Embedding

A Grassmannian variety or Grassmannian (denoted $G(k, V)$) is a projective variety whose closed points correspond to the vector subspaces of a certain dimension k in a given vector space V.

$G(2,4)$ then is the set of vector subspaces of dimension 2 within $\mathbb{R}^4$, equivalently, is the set of projective lines $\mathbb{P}^1$ in $\mathbb{P}^3$. The Plücker embedding

$$\psi : G(2,4) \longrightarrow \mathbb{P}(\mathbb{C}^6) = \mathbb{P}^5$$

is a map sending V in $G(2,4)$ to $[p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34}]$ where p_{ij} is the determinant of the $(2 \times 2)-$sub-matrix consisted of i and j-th columns of the matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}$ for a basis $\{\mathbf{v}_1 = (a_{11}, a_{12}, a_{13}, a_{14}), \mathbf{v}_2 = (a_{21}, a_{22}, a_{23}, a_{24})\}$ for V.

The Plücker embedding maps $G(2,4)$ to a quadric hypersurface in $\mathbb{P}^5$.

$$\psi(G(2,4)) = V(z_0 z_5 - z_1 z_4 + z_2 z_3) \subset \mathbb{P}^5$$

Images of the Families of lines on Q Under the Embedding

A line ℓ in each of the families $\mathcal{F}_i$ of a quadric surface Q is an element in $G(2,4)$, and therefore it can be mapped under the embedding.

$$\psi(\mathcal{F}_1) = V({z_5}^2 + z_1 z_4) \subset V(z_2, z_3, z_0 + z_5) \cong \mathbb{P}^2$$
$$\psi(\mathcal{F}_2) = V({z_5}^2 + z_2 z_3) \subset V(z_1, z_4, z_0 - z_5) \cong \mathbb{P}^2$$

In other words, the families of lines are mapped to projective plane conics on the quadric hysurface $\psi(G(2,4))$. Specifically, they are two conics that lie in complementary planes that do not intersect, abandoning its aforementioned properties.

References

- Baez, J. (2016, February 15). 27 Lines on a Cubic Surface. American Mathematical Society Blogs.
- Egan, G. 27 Lines on a Cubic Surface.