Skeletal 2-Groups and Category Theory

Nicholas Small, Alexander Stepanov, Haley Waiksnis

Providence College

Historical Background

Category theory is an unifying perspective on structure that lies at the foundation of many diverse areas of mathematics. It is concerned about things (objects) and relationships between them (morphisms)... and all the structure that one can describe and unpack about these abstract relationships. It complements set theory (a traditional foundation for much of mathematics), by looking beyond elements of a set to also study the *relationships* between these elements.

In the 1970s, Hoàng Xuân Sính, the first female mathematics professor in Vietnam, wrote a thesis about *Gr-categories* or *2-groups* [2, 1]. In her work, Hoàng looks at symmetries, and symmetries of symmetries: just the type of relationships, and relationships between relationships (and so on...) that category theory is suited to describe.

Category Theory Background

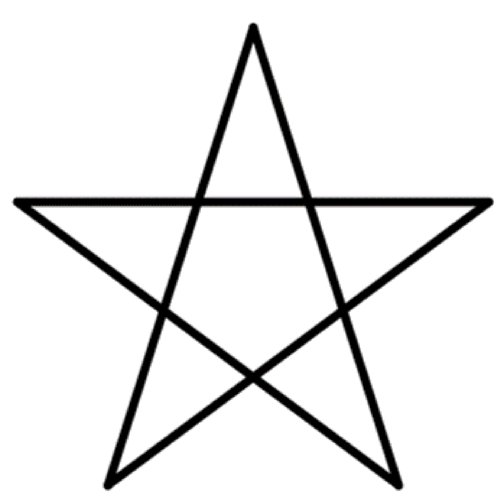
Groups

A group is, in the simplest terms, a set G and an operation

$$*: G \times G \to G,$$

that satisfies the following properties:

- $*$ is an associative operation; for all $a, b, c \in G$, $(a * b) * c = a * (b * c)$.
- There exists an $e \in G$ that is the identity element; for all $a \in G, a * e = a = e * a$.
- Every element has an inverse; for every $a \in G$, there exists an $a^{-1} \in G$ such that $a * a^{-1} = e = a^{-1} * a$.



Categories

A category builds on the idea of a set, but puts equal emphasis on the elements of a set and the *relationships* between them. Categories have two structures:

- the objects, obj$\mathcal{C}$
- the morphisms: $A, B \in$ obj$\mathcal{C}$, we have the collection of morphisms $f : A \to B \in \text{Hom}_{\mathcal{C}}(A, B)$. Compatible morphisms can be composed.

Morphisms have some constraints:

- Associativity: If $A, B, C, D \in$ obj$\mathcal{C}$, $f : A \to B, g : B \to C, h : C \to D$, then $(f \circ g) \circ h = f \circ (g \circ h)$.
- Unitality: If $A, B \in$ obj$\mathcal{C}$, $f : A \to B$, then there exists $1_A : A \to A$ and $1_B : B \to B$ such that $f \circ 1_A = f = 1_B \circ f$.

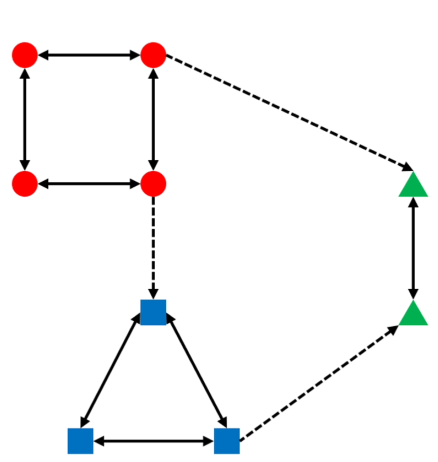


Figure 1 A graphical representation of a category, where the shapes represent objects and the arrows represent morphisms.

Types of Categories

Monoidal Categories

A monoidal category introduces group-like structure to a category. It consists of a category $\mathcal{C}$ with the following additional structure:

$$\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

(on objects) $\quad (g_1, g_2) \longmapsto g_1 \otimes g_2$

(on morphisms) $\quad g_1, g_2 \longmapsto g_1 \otimes g_2$, with $\phi: g_1 \to g_1'$, $\psi: g_2 \to g_2'$ and $\phi \otimes \psi : g_1 \otimes g_2 \to g_1' \otimes g_2'$

- There is a unit object I; for any object $g, g \otimes I \xrightarrow{\cong} g \xleftarrow{\cong} I \otimes g$.
- There is an associator isomorphism α:

$$\alpha_{a,b,c} : (a \otimes b) \otimes c \xrightarrow{\cong} a \otimes (b \otimes c).$$

α must also satisfy the pentagon axiom (shown below).

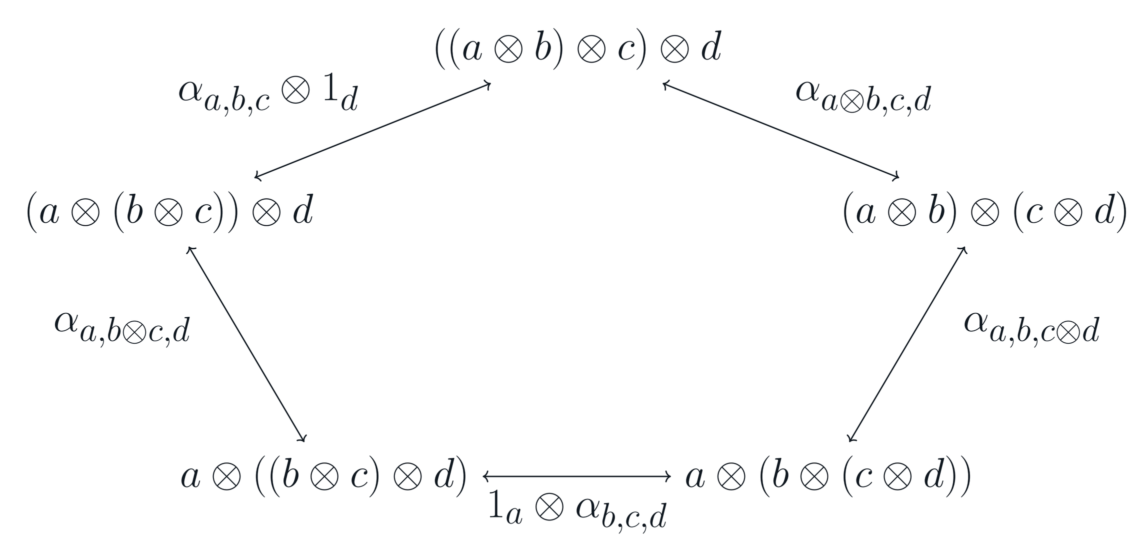


Figure 2 A diagram of the pentagon axiom for $a, b, c, d \in$ obj$\mathcal{C}$.

A *2-group* is a monoidal category in which all the objects and morphisms have inverses.

Skeletal Categories

Given a category $\mathcal{C}$, there's a simpler, equivalent category one can consider: the skeleton of $\mathcal{C}$. This skeletal category is created by selecting a single representative object from each isomorphism cluster in the category, while maintaining the morphisms between each element.

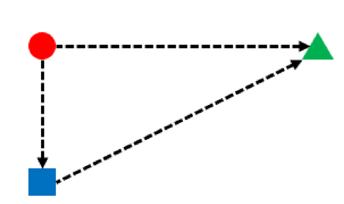


Figure 3 The skeletal category of **Figure 1**.

Classification Result

For this project, we were interested in classifying 2-groups (up to equivalence). We first simplified the problem by looking at skeletal 2-groups, since any 2-group is equivalent to its skeletal category. Such skeletal 2-groups can be described in terms of the following data:

- Objects: a group G
- Morphisms: an abelian group $A := \text{Aut}_{\mathcal{C}}(I)$
  - a group homomorphism $\rho : \mathcal{C} \to \text{Aut}(A)$

Cocyles, Coboundaries and Group Cohomology

The associator α gives rise to a map $a : G^3 \to A$, and the pentagon axiom that α satisfies gives the following condition on a for any $(g_0, g_1, g_2) \in G^3$:

$$da = \rho(g_0) \cdot a(g_1, g_2) - a(g_0 \otimes g_1, g_2) + a(g_0, g_1 \otimes g_2) - a(g_1, g_2).$$

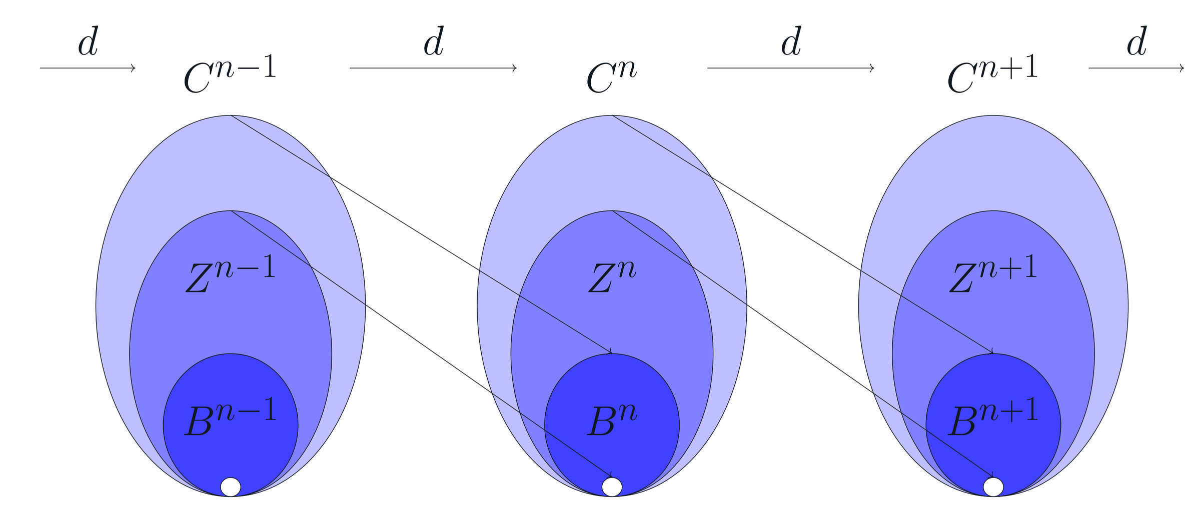
Cocycles, Coboundaries and Group Cohomology

Let $C^n = \{f : G^n \to A\}$; the elements of this group are called *cochains*. Consider the following sequence of maps:

$$C^1 \xrightarrow{d} C^2 \xrightarrow{d} C^3 \xrightarrow{d} C^4 \xrightarrow{d} \cdots C^{n-1} \to C^n \xrightarrow{d} \cdots$$

where the maps $d : C^{n-1} \to C^n$ are defined in an analogous way to the above equation. We looked at two subgroups of C^n:

- $Z^n = \ker(d) = \{f : G^n \to A \,|\, df = 0\}$ or the n-*cocycles*
- $B^n = \text{im}(d) = \{g : G^n \to A \,|\, df = g$ for some $f \in C^{n-1}\}$ or the n-*coboundaries*



Note that for any $f \in C^n$, $d^2 f = 0$. It then makes sense to consider the quotient group. A quotient group is the set of cosets of a normal subgroup of a group. In this case $H^n(G; A) := Z^n / B^n$ is called the n^{th} cohomology group.

Example: $G = \mathbb{Z}_n, A = \mathbb{C}^\times$

In the particular case where $G = \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} = \{[x]_n \,|\, x \in \mathbb{Z}, 0 \leq x < n\}$, the cyclic group, we can define the group $H^3(\mathbb{Z}_n^3, \mathbb{C}^\times)$. Here $H^3(\mathbb{Z}_n^3, \mathbb{C}^\times) \cong \mathbb{Z}_n$.

For $[\phi_k] \in H^3(\mathbb{Z}_n, \mathbb{C}^\times)$, we defined a specific representative cocyle $\phi_k : \mathbb{Z}_n^3 \to \mathbb{C}^\times$ such that

$$\phi_k(a, b, c) = \begin{cases} 1 & b + c < n \\ e^{\frac{2i\pi ka}{n}} & b + c \geq n \end{cases}.$$

Acknowledgements

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References

[1] John C. Baez. "Hoàng Xuân Sính's Thesis: Categorifying Group Theory". In: vol. 2. 2023, pp. 1–35. DOI: https://math.ucr.edu/home/baez/sinh.pdf.

[2] Xuân Sính Hoàng. "Gr-categories, Université Paris VII". PhD thesis. PhD thesis, 1975.